Replica calculations for the SK model

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1 Sherrington-Kirkpatrick model

Consider the Hamiltonian

$$H = \sum_{i<j} J_{ij} S_i S_j - h \sum_i S_i \tag{1}$$

where the bonds J_{ij} are quenched variables following a Gaussian distribution

$$P(J_{ij}) = \frac{1}{J}\sqrt{\frac{N}{2\pi}} \exp(-\frac{N}{2J^2}\left(J_{ij} - \frac{J_0}{N}\right)^2). \tag{2}$$

Note that its mean is $\frac{J_0}{N}$ and variance $\frac{J^2}{N}$. It is so defined such that some quantities of the system are proportional to N, which is physical.

Here J_{ij} and S_i are both random variables. We assume the former to be quenched, i.e. fixed on the timescale on which the latter fluctuates. Further, we define the average of some quantity over the S_i the THERMAL AVERAGE and denote with $< \cdot >$; the average over J_{ij} is the CONFIGURATIONAL AVERAGE and denote with $[\cdot]$.

Let $\boldsymbol{s} = \{S_1, S_2, ..., S_N\}$. We are interested in the configurational average of the free energy, defined as

$$[F(\boldsymbol{s})] = -T[\log Z(\boldsymbol{s})] = -T[\log \sum_{\{\boldsymbol{s}\}} \exp(-\beta H(\boldsymbol{s}))]. \tag{3}$$

Here the sum is over all possible values of $\boldsymbol{s}$. This corresponds to the Tr operator in the Nishimori text.

2 Calculating the free energy with replica method

To compute $[F]$, we use the exact identity

$$[\log Z] = \lim_{n\to 0} \frac{[Z^n] - 1}{n}. \tag{4}$$

Plug in $Z = e^{-\beta H}$ and Eq 1, write $[Z^n]$

$$[Z^n] = \int \prod_{i<j} dJ_{ij} P(J_{ij}) \sum_{\{\boldsymbol{s}^\alpha, \boldsymbol{s}^\beta, ..., \boldsymbol{s}^n\}} \exp\left(\beta \sum_{i<j} J_{ij} \sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha + \beta h \sum_{i=1}^{N} \sum_{\alpha=1}^{n} S_i^\alpha\right) \tag{5}$$

Now, for the variables $\{J_{ij}\}$, plug in expressions for $P(J_{ij})$ and complete the squares in the exponents to get [1]

$$[Z^n] = C_1 \sum_{\{\boldsymbol{s}^\alpha, \boldsymbol{s}^\beta, ..., \boldsymbol{s}^n\}} \exp\left\{\frac{1}{N}\sum_{i<j}\left(\frac{1}{2}\beta^2 J^2 \sum_{\alpha,\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \beta J_0 \sum_\alpha S_i^\alpha S_j^\alpha\right) + \beta h \sum_{i=1}^{N}\sum_{\alpha=1}^{n} S_i^\alpha\right\}. \tag{6}$$

The constant factor does not depend on $\{\boldsymbol{s}, \boldsymbol{J}, N\}$. This step is the key usefulness of the replica method. We simply integrated out the J_{ij} and obtained an expression that does not depend on them.

The last term is not changed because it does not contain J_{ij}. We now rewrite several things. First, $\sum_{\alpha,\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta = 2\sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \sum_\alpha \left(S_i^\alpha S_j^\alpha\right)^2 = 2\sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + n$. Therefore

[1] Superscripts with α, β are not exponents.

$$[Z^n] = C_1 \sum_{\{s^\alpha, s^\beta, \dots, s^n\}} \exp\left\{\frac{1}{N}\sum_{i<j}\left(\frac{1}{2}\beta^2 J^2\left(2\sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + n\right) + \beta J_0 \sum_\alpha S_i^\alpha S_j^\alpha\right) + \beta h \sum_{i=1}^N \sum_{\alpha=1}^n S_i^\alpha\right\} \tag{7}$$

$$= C_1 \sum_{\{s^\alpha, s^\beta, \dots, s^n\}} \exp\left\{\frac{1}{N}\sum_{i<j}\left(\frac{1}{2}\beta^2 J^2 n + \frac{1}{2}\beta^2 J^2\left(2\sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta\right) + \beta J_0 \sum_\alpha S_i^\alpha S_j^\alpha\right) + \beta h \sum_{i=1}^N \sum_{\alpha=1}^n S_i^\alpha\right\} \tag{8}$$

$$= C_1 \exp\left(\frac{(N-1)\,\beta^2 J^2 n}{4}\right) \sum_{\{s^\alpha, s^\beta, \dots, s^n\}} \exp\left\{\frac{1}{N}\sum_{i<j}\left(\beta^2 J^2 \sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \beta J_0 \sum_\alpha S_i^\alpha S_j^\alpha\right) + \beta h \sum_{i=1}^N \sum_{\alpha=1}^n S_i^\alpha\right\} \tag{9}$$

To get the first factor, we used the fact that $\sum_{i<j}$ has $\frac{N(N-1)}{2}$ terms. Further, since we assume a large N, $\exp\left(\frac{(N-1)\beta^2 J^2 n}{4}\right) \approx \exp\left(\frac{N\beta^2 J^2 n}{4}\right)$. Now we look at the second exponent (after Tr). First,

$$\frac{1}{N}\sum_{i<j}\beta^2 J^2 \sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta = \frac{\beta^2 J^2}{2N}\left(\sum_{\alpha<\beta}\left(\sum_i S_i^\alpha S_i^\beta\right)^2 - \sum_i \sum_{\alpha<\beta} S_i^\alpha S_i^\alpha S_i^\beta S_i^\beta\right)$$
$$= \frac{\beta^2 J^2}{2N}\sum_{\alpha<\beta}\left(\sum_i S_i^\alpha S_i^\beta\right)^2 - \frac{\beta^2 J^2}{2N}\sum_i \sum_{\alpha<\beta} 1.$$

We shall neglect the last term, since it is constant in $\{\boldsymbol{s}, \boldsymbol{J}, N\}$ and simply leads to a coefficient. The same can be done for the second term. I.e.

$$\frac{\beta J_0}{N}\sum_{i<j}\sum_\alpha S_i^\alpha S_j^\alpha = \frac{\beta J_0}{2N}\sum_\alpha\left(\sum_i S_i^\alpha\right)^2 - \frac{\beta J_0}{2N}\sum_i\sum_\alpha 1,$$

where the last term is neglected for the same reason. Combining all the transformations, we now have

$$[Z^n] = C_2 \exp\left(\frac{N\beta^2 J^2 n}{4}\right) \sum_{\{s^\alpha, s^\beta, \dots, s^n\}} \exp\left\{\frac{\beta^2 J^2}{2N}\sum_{\alpha<\beta}\left(\sum_i S_i^\alpha S_i^\beta\right)^2 + \frac{\beta J_0}{2N}\sum_\alpha\left(\sum_i S_i^\alpha\right)^2 + \beta h \sum_{i=1}^N \sum_{\alpha=1}^n S_i^\alpha\right\}. \tag{10}$$

Note that the right hand side here has a different coefficient from that in the RHS in Eq. 9.

The goal now is to "linearize" the quadratic terms[2] in Eq.10 with the following identity (this is sometimes referred to as the Hubbard-Stratonovich transform)

$$\exp\left(\frac{y^2}{2}\right) = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\exp\left(xy\right). \tag{11}$$

In this identity, we introduce a normally distributed variable x to linearize the quadratic term y^2. For Eq.10, we introduce a $Nq_{\alpha\beta}$ for $\left(\sum_i S_i^\alpha S_i^\beta\right)^2$ and Nm_α for $\sum_{\alpha=1}^n S_i^\alpha$. Then, by the identity above,

$$\exp\frac{\beta^2 J^2}{2N}\left(\sum_i S_i^\alpha S_i^\beta\right)^2 = \int_{-\infty}^{\infty}\frac{dNq_{\alpha\beta}}{\sqrt{2\pi}}\exp\left(-\beta^2 J^2 N\frac{q_{\alpha\beta}^2}{2} + \beta^2 J^2 q_{\alpha\beta}\sum_i S_i^\alpha S_i^\beta\right) \tag{12}$$

$$\exp\frac{\beta J_0}{2N}\left(\sum_i S_i^\alpha\right)^2 = \int_{-\infty}^{\infty}\frac{dNm_\alpha}{\sqrt{2\pi}}\exp\left(-\beta J_0 N m_\alpha^2 + \beta J_0 m_\alpha \sum_i S_i^\alpha\right). \tag{13}$$

Do this for all the $q_{\alpha\beta}$, m_α, and absorb the resulting $\sqrt{2\pi}$ into the constant coefficient, we have

[2]This is desirable because if we have $\sum_{\{s\}}(\exp(a_1 s_1 + a_2 s_2 \dots))$, then we can rewrite it as $\sum_{s_1=\pm1}\exp(a_1 s_1)\sum_{s_2=\pm1}\exp(a_1 s_2)\dots$.

$$
\begin{aligned}
[Z^n] = C_3 \exp\left(\frac{N\beta^2 J^2 n}{4}\right) \int_{-\infty}^{\infty} \prod_{\alpha<\beta} dNq_{\alpha\beta} \prod_{\alpha} dNm_\alpha \cdot \\
\sum_{\{\boldsymbol{s}^\alpha, \boldsymbol{s}^\beta, \ldots, \boldsymbol{s}^n\}} \exp\left(-\frac{\beta^2 J^2 N}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta J_0 N}{2} \sum_\alpha m_\alpha^2\right) \cdot \\
\exp\left(\beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta + \beta J_0 \sum_\alpha m_\alpha \sum_i S_i^\alpha\right) \exp\left(\beta h \sum_{i=1}^N \sum_{\alpha=1}^n S_i^\alpha\right).
\end{aligned}
$$

We can integrate over $q_{\alpha\beta}, m_\alpha$ instead of $Nq_{\alpha\beta}$,Nm_α by multiplying everything by the Jacobians, which is a power of N. Absorb it again into the coefficient to get

$$
[Z^n] = C_4 \exp\left(\frac{N\beta^2 J^2 n}{4}\right) \int_{-\infty}^{\infty} \prod_{\alpha<\beta} dq_{\alpha\beta} \prod_{\alpha} dm_\alpha \cdot \tag{14}
$$

$$
\exp\left(-\frac{\beta^2 J^2 N}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta J_0 N}{2} \sum_\alpha m_\alpha^2\right) \cdot \tag{15}
$$

$$
\sum_{\{\boldsymbol{s}^\alpha, \boldsymbol{s}^\beta, \ldots, \boldsymbol{s}^n\}} \exp\left(\beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta + \beta \sum_\alpha (J_0 m_\alpha + h) \sum_i S_i^\alpha\right). \tag{16}
$$

For the last term, rewrite it as

$$
\prod_{i=1}^N \sum_{\{s_i^\alpha, s_i^\beta, \ldots, s_i^n)} \exp\left(\beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} S_i^\alpha S_i^\beta + \beta \sum_\alpha (J_0 m_\alpha + h) S_i^\alpha\right). \tag{17}
$$

Since for every i we are summing it over the same variables $\{s_i^\alpha, s_i^\beta, ..., s_i^n)$ over the same values (± 1), and there's no dependence on i otherwise, all N terms in the product are the same. Using S^α to denote a "general site" on the lattice, we rewrite this as

$$
\left\{\sum_{\{s_i^\alpha, s_i^\beta, \ldots, s_i^n)} \exp\left(\beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} S^\alpha S^\beta + \beta \sum_\alpha (J_0 m_\alpha + h) S^\alpha\right)\right\}^N \equiv \exp\left\{N \log \sum_{\{s_i^\alpha, s_i^\beta, \ldots, s_i^n)} \exp\left(L\left(\{q_{\alpha\beta}, m_\alpha\}\right)\right)\right\} \tag{18}
$$

where

$$
\boxed{L\left(\{q_{\alpha\beta}, m_\alpha\}\right) := \beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} S^\alpha S^\beta + \beta \sum_\alpha (J_0 m_\alpha + h) S^\alpha}. \tag{19}
$$

We thus obtain

$$
[Z^n] = C_4 \exp\left(\frac{N\beta^2 J^2 n}{4}\right) \int_{-\infty}^{\infty} \prod_{\alpha<\beta} dq_{\alpha\beta} \prod_{\alpha} dm_\alpha \tag{20}
$$

$$
\exp\left\{-\frac{\beta^2 J^2 N}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta J_0 N}{2} \sum_\alpha m_\alpha^2 + N \log \sum_{\{S^\alpha, S^\beta, \ldots, S^n)} \exp(L)\right\}. \tag{21}
$$

Importantly, the exponent is proportional to N. At the large N limit, value of the integral is determined by maximum of the integral (METHOD OF STEEPEST DESCENT). Let

$$
E := -\frac{\beta^2 J^2 N}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta J_0 N}{2} \sum_\alpha m_\alpha^2 + N \log \sum_{\{S^\alpha, S^\beta, \ldots, S^n\}} \exp(L) \tag{22}
$$

$$
\{q_{\alpha\beta}^\star, m_\alpha^\star\} = \arg \max_{\{q_{\alpha\beta}, m_\alpha\}} E. \tag{23}
$$

The integral is

$$[Z^n] = C_4 \exp\left\{\frac{N\beta^2J^2n}{4} - \frac{\beta^2J^2N}{2}\sum_{\alpha<\beta}\left(q^\star_{\alpha\beta}\right)^2 - \frac{\beta J_0 N}{2}\sum_\alpha \left(m^\star_\alpha\right)^2 + N\log \sum_{\{S^\alpha,S^\beta,\ldots,S^n\}} \exp(L)\right\} \tag{24}$$

$$= C_4 \exp\left\{Nn\left(\frac{\beta^2J^2}{4} - \frac{\beta^2J^2}{2n}\sum_{\alpha<\beta}\left(q^\star_{\alpha\beta}\right)^2 - \frac{\beta J_0}{2n}\sum_\alpha \left(m^\star_\alpha\right)^2 + \frac{1}{n}\log \sum_{\{S^\alpha,S^\beta,\ldots,S^n\}} \exp(L)\right)\right\}. \tag{25}$$

Now as $n \to \infty$, the exponent goes to zero. We expand the exponent around 0 to get

$$[Z^n] \approx 1 + Nn\left\{\frac{\beta^2J^2}{4} - \frac{\beta^2J^2}{2n}\sum_{\alpha<\beta}\left(q^\star_{\alpha\beta}\right)^2 - \frac{\beta J_0}{2n}\sum_\alpha \left(m^\star_\alpha\right)^2 + \frac{1}{n}\log \sum_{\{S^\alpha,S^\beta,\ldots,S^n\}} \exp(L)\right\}. \tag{26}$$

We are finally ready to plug it back into the replica identity to get

$$\frac{[\log Z]}{N} = \lim_{n\to 0}\frac{[Z^n]-1}{Nn} \tag{27}$$

$$= \lim_{n\to 0}\left\{\frac{\beta^2J^2}{4} - \frac{\beta^2J^2}{2n}\sum_{\alpha<\beta}\left(q^\star_{\alpha\beta}\right)^2 - \frac{\beta J_0}{2n}\sum_\alpha \left(m^\star_\alpha\right)^2 + \frac{1}{n}\log \sum_{\{S^\alpha,S^\beta,\ldots,S^n\}} \exp(L)\right\}. \tag{28}$$

We now examine $\{q^\star_{\alpha\beta}, m^\star_\alpha\}$ more carefully. A necessary condition for them to maximize the exponent in Eq.22 is that $\frac{\partial}{\partial q_{\alpha\beta}}E = \frac{\partial}{\partial m_\alpha}E = 0$. I.e.

$$q^\star_{\alpha\beta} = \frac{1}{\beta^2J^2}\frac{\partial}{\partial q_{\alpha\beta}}\log \sum_{\{S^\alpha,S^\beta,\ldots,S^n\}} \exp(L) = \frac{1}{\beta^2J^2}\frac{\sum_{\{S^\alpha,S^\beta,\ldots,S^n\}}\exp(L)\,\beta^2J^2}{\sum_{\{S^\alpha,S^\beta,\ldots,S^n\}}\exp(L)}S^\alpha S^\beta. \tag{29}$$

$$= \frac{\sum_{\{S^\alpha,S^\beta,\ldots,S^n\}} S^\alpha S^\beta \exp(L)}{\sum_{\{S^\alpha,S^\beta,\ldots,S^n\}}\exp(L)} \tag{30}$$

$$m^\star_\alpha = \frac{1}{\beta J_0}\frac{\partial}{\partial m_\alpha}\log \sum_{\{s_i^\alpha,s_i^\beta,\ldots,s_i^n)} \exp(L) = \frac{\sum_{\{S^\alpha,S^\beta,\ldots,S^n\}} S^\alpha \exp(L)}{\sum_{\{S^\alpha,S^\beta,\ldots,S^n\}}\exp(L)}. \tag{31}$$

Whether these represent the maximum needs to be determined from the second derivative. This issue is visited later.

2.1 $q_{\alpha\beta}$ and m_α as order parameters (TODO: check these identities)

It can be confirmed that

$$q_{\alpha\beta} = \left[\frac{\sum_{\{\boldsymbol{s}^\alpha,\boldsymbol{s}^\beta,\ldots,\boldsymbol{s}^n\}} S_i^\alpha S_i^\beta \exp\left(-\beta\sum_{\gamma=1}^n H_\gamma\right)}{\sum_{\{\boldsymbol{s}^\alpha,\boldsymbol{s}^\beta,\ldots,\boldsymbol{s}^n\}} \exp\left(-\beta\sum_{\gamma=1}^n H_\gamma\right)}\right] \equiv \left[\left\langle S_i^\alpha S_i^\beta\right\rangle\right] \tag{32}$$

where

$$H_\gamma = \sum_{i<j} J_{ij}S_i^\gamma S_j^\gamma - h\sum_i S_i^\gamma \tag{33}$$

and

$$m_\alpha = [\langle S_i^\alpha\rangle]\,. \tag{34}$$

3 Replica-symmetric solution

How do $q_{\alpha\beta}$ and m_α depend on α, β? A naive guess is that, since all replicas are equivalent, at the end of the day we should have

$$\forall \alpha,\beta : q_{\alpha\beta} = q, m_\alpha = m.$$

This is the REPLICA-SYMMETRIC SOLUTION. Plug these expressions into that for the free energy, we have

$$\frac{[\log Z]}{N} = \lim_{n\to 0}\left\{\frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2 (n-1)}{4} q^2 - \frac{\beta J_0}{2} m^2 + \frac{1}{n}\log \sum_{\{S^\alpha, S^\beta, \ldots, S^n\}} \exp\left(L_{q_{\alpha\beta}=q, m_\alpha = m}\right)\right\}. \tag{35}$$

$$= \frac{\beta^2 J^2}{4}\left(1+q^2\right) - \frac{\beta J_0}{2} m^2 + \lim_{n\to 0}\frac{1}{n}\log \sum_{\{S^\alpha, S^\beta, \ldots, S^n\}} \exp\left(L_{q_{\alpha\beta}=q, m_\alpha = m}\right). \tag{36}$$

For the last term, plugging in q, m

$$\frac{1}{n}\log \sum_{\{s_i^\alpha, s_i^\beta, \ldots, s_i^n)} \exp\left(L_{q_{\alpha\beta}=q, m_\alpha = m}\right) = \log \sum_{\{S^\alpha, S^\beta, \ldots, S^n\}} \exp\left(\beta^2 J^2 q \sum_{\alpha<\beta} S^\alpha S^\beta + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha\right). \tag{37}$$

We can introduce a Gaussian variable to linearize the product $S^\alpha S^\beta$

$$p\left(\hat{z}\right) = \sqrt{\frac{\beta^2 J^2 q}{2\pi}} \exp\left(-\frac{\hat{z}^2}{2}\beta^2 J^2 q\right) \tag{38}$$

and rewrite (using the same identity from Eq.11)

$$\begin{aligned}
&\frac{1}{n}\log \sum_{\{s_i^\alpha, s_i^\beta, \ldots, s_i^n)} \exp\left(\beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} S^\alpha S^\beta + \beta \sum_\alpha \left(J_0 m_\alpha + h\right) S^\alpha\right) \\
=&\frac{1}{n}\log \sum_{\{s_i^\alpha, s_i^\beta, \ldots, s_i^n)} \int dz p\left(z\right) \exp\left(\beta^2 J^2 q \hat{z} \sum_\alpha S^\alpha - \frac{n}{2}\beta^2 J^2 q + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha\right)
\end{aligned}$$

The $-\frac{n}{2}\beta^2 J^2 q$ term came from $\left(\sum_\alpha S^\alpha\right)^2 - 2\sum_{\alpha<\beta} S^\alpha S^\beta = n$. Further,

$$\begin{aligned}
&\frac{1}{n}\log \int d\hat{z} p\left(\hat{z}\right) \sum_{\{s_i^\alpha, s_i^\beta, \ldots, s_i^n)} \exp\left(\beta^2 J^2 q \hat{z} \sum_\alpha S^\alpha + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha - \frac{n}{2}\beta^2 J^2 q\right) \\
=&\frac{1}{n}\log \exp\left(-\frac{n}{2}\beta^2 J^2 q\right) \int dz p\left(\hat{z}\right) \sum_{\{S^\alpha, S^\beta, \ldots, S^n\}} \exp\left(\sum_\alpha S^\alpha \left(\beta^2 J^2 q \hat{z} + \beta\left(J_0 m + h\right)\right)\right)
\end{aligned}$$

Define $z \sim \mathcal{N}\left(0,1\right)$ and Dz to be the standard Gaussian measure. Then we perform a change of variable and obtain

$$\frac{1}{n}\log \exp\left(-\frac{n}{2}\beta^2 J^2 q\right) \int Dz \sum_{\{S^\alpha, S^\beta, \ldots, S^n\}} \exp\left(\sum_\alpha S^\alpha \left(\beta J \sqrt{q} z + \beta\left(J_0 m + h\right)\right)\right)$$

Now that the exponent is linear in S^α, we can perform the summation separately for each replica, i.e.

$$\begin{aligned}
&\sum_{\{s_i^\alpha, s_i^\beta, \ldots, s_i^n)} \exp\left(\sum_\alpha S^\alpha \left(\beta J \sqrt{q} z + \beta\left(J_0 m + h\right)\right)\right) \\
=&\prod_{\gamma=1}^{n} \sum_{S=\pm 1} \exp\left(S\left(\beta J \sqrt{q} z + \beta\left(J_0 m + h\right)\right)\right) \\
=&\left\{2\cosh\left(\beta J \sqrt{q} z + \beta\left(J_0 m + h\right)\right)\right\}^n
\end{aligned}$$

Hence the last term in Eq.36 can be written as

$$\frac{1}{n}\log \int Dz \exp\left(n \log 2\cosh\left(\beta J \sqrt{q} z + \beta\left(J_0 m + h\right)\right) - \frac{n}{2}\beta^2 J^2 q\right). \tag{39}$$

We now take the small n limit, and expand the exponential around 0

$$\approx \frac{1}{n} \log \left\{ 1 + n \int Dz \log 2 \cosh \left(\beta J \sqrt{q} z + \beta \left(J_0 m + h \right) \right) - \frac{n}{2} \beta^2 J^2 q \right\} \tag{40}$$

Expand the logarithm around 1 to get

$$\approx \int Dz \log 2 \cosh \left(\beta J \sqrt{q} z + \beta \left(J_0 m + h \right) \right) - \frac{n}{2} \beta^2 J^2 q. \tag{41}$$

Define

$$\tilde{H}(z) := J\sqrt{q} z + J_0 m + h \tag{42}$$

and obtain

$$\lim_{n \to 0} \frac{1}{n} \log \sum_{\{s_i^\alpha, s_i^\beta, \ldots, s_i^n\}} \exp \left(L_{q_{\alpha\beta}=q, m_\alpha = m} \right) = \int Dz \log 2 \cosh \left(\beta \tilde{H}(z) \right) - \frac{1}{2} \beta^2 J^2 q. \tag{43}$$

Plug this back into Eq.36, we have a final expression for the free energy in terms of q, m

$$\frac{[\log Z]}{N} = \frac{\beta^2 J^2}{4} (1-q)^2 - \frac{\beta J_0}{2} m^2 + \int Dz \log 2 \cosh \left(\beta \tilde{H}(z) \right). \tag{44}$$

We now set $\partial_q \frac{[\log Z]}{N} = \partial_m \frac{[\log Z]}{N} = 0$ to obtain

$$m = \int Dz \tanh \beta \tilde{H}(z) \tag{45}$$

$$q = 1 - \int Dz \mathrm{sech}^2 \beta \tilde{H}(z) = \int Dz \tanh^2 \beta \tilde{H}(z). \tag{46}$$

4 Replica symmetry breaking and the Parisi solution

4.1 Problem with the symmetric results: negative entropy at low temperature

Here we assume $J_0 = h = 0$. Thus $\tilde{H}(z) = J\sqrt{q} z$.

According to Eq.46, as $T \to 0$, or $\beta \to \infty$, $q \to 1$. We therefore guess that near this limit, q can be linearized around $T = 0$ with an unknown factor a. I.e., $q = 1 - aT, a > 0$. Then,

$$\lim_{\beta \to \infty} \int Dz \mathrm{sech}^2 \beta \tilde{H}(z) = \frac{1}{\beta J} \int Dz \left(\frac{d}{dz} \tanh \beta J z \right) = \frac{1}{\beta J} Dz \left(2\delta(z) \right) = \sqrt{\frac{2}{\pi}} \frac{T}{J}. \tag{47}$$

The unknown $a = \sqrt{\frac{2}{\pi}}/J$. To compute the entropy, we first compute the free energy at this limit. Under the assumption of $J_0 = h = 0$, Eq.44 becomes

$$\beta [f] = -\frac{[\log Z]}{N} = -\frac{\beta^2 J^2}{4} (1-q)^2 - \int Dz \log 2 \cosh \left(\beta J \sqrt{q} z \right). \tag{48}$$

Here $[f]$ is the configurational average of the per-spin free energy.

Plugging in $q = 1 - T\sqrt{\frac{2}{\pi}}/J$, the first term gives $-\frac{T}{2\pi}$. For the second term[3], since the integrand is even and assuming large β

$$\begin{aligned}
\int Dz \log 2 \cosh \left(\beta J \sqrt{q} z \right) &= 2 \int_0^\infty Dz \log 2 \cosh \left(\beta J \sqrt{q} z \right) && (49) \\
&= 2 \int_0^\infty Dz \log 2 \frac{\exp\left(-\beta J \sqrt{q} z \right) + \exp\left(\beta J \sqrt{q} z \right)}{2} && (50) \\
&\approx 2 \int_0^\infty Dz \log 2 \frac{\exp\left(\beta J \sqrt{q} z \right)}{2} \approx 2\beta J \sqrt{q}. && (51)
\end{aligned}$$

Expand $\sqrt{q}$ around $q = 1$ to get

[3] This derivation is slightly different from the one presented in the book.

$$\int Dz \log 2\cosh\left(\beta J\sqrt{q}z\right) \approx 2\beta J\left(2\pi\right)^{-1/2}\left(1-aT/2\right). \tag{52}$$

Adding the two terms together to get

$$[f] \approx -\sqrt{\frac{2}{\pi}}J + \frac{T}{2\pi}. \tag{53}$$

Since the $F = E - TS$, we conclude that

$$S = -\frac{1}{2\pi}. \tag{54}$$

The negative entropy is inconsistent with the fact that SK models have discrete degrees of freedom (and therefore have non-negative entropies). It was later found that this came from the assumption of replica symmetry, not the numerous sketchy math steps that we performed[4].

4.2 Stability of solutions

When deriving the free energy, we used the Method of Steepest Descent. It only works if we can maximize the exponent. In the derivations above, we only extremized it w.r.t. $q_{\alpha\beta}, m_\alpha$. To see whether we found a maximum, we need to see whether the Hessian is positive definite. *We again assume* $h = 0$.

Denote

$$y^{\alpha\beta} := \beta J q_{\alpha\beta}, x^\alpha = \sqrt{\beta J_0} m_\alpha. \tag{55}$$

Rewrite Eq.28 as

$$[f] = -\frac{\beta J^2}{4} - \lim_{n\to 0}\frac{1}{\beta n}\left\{-\sum_{\alpha<\beta}\frac{1}{2}\left(y^{\alpha\beta}\right)^2 - \sum_\alpha \frac{1}{2}\left(x^\alpha\right)^2 + \log\sum_{\{S^\alpha,S^\beta,\dots,S^n\}}\exp\left(\beta J\sum_{\alpha<\beta}y^{\alpha\beta}S^\alpha S^\beta + \sqrt{\beta J_0}\sum_\alpha x^\alpha S^\alpha\right)\right\}. \tag{56}$$

Write $x^\alpha = x + \epsilon^\alpha$, $y^{\alpha\beta} = y + \eta^{\alpha\beta}$. We expand all the $x^\alpha, y^{\alpha\beta}$ around the same value because that was the assumption of replica symmetry. Then write

$$L_0 := \beta J y \sum_{\alpha<\beta} S^\alpha S^\beta + \sqrt{\beta J_0} x \sum_\alpha S^\alpha \tag{57}$$

$$\langle f\rangle_{L^0} = \frac{\sum_{\{S^\alpha,S^\beta,\dots,S^n\}} e^{L_0\left(\{S^\alpha,S^\beta,\dots,S^n\}\right)} f}{\sum_{\{S^\alpha,S^\beta,\dots,S^n\}} e^{L_0(\{S^\alpha,S^\beta,\dots,S^n\})}}. \tag{58}$$

Then we can start expanding $[f]$ to the second order. Remember that, since we've already extremized $[f]$ w.r.t. $q^{\alpha\beta}, m^\alpha$, the first order derivatives are all zero. Also, $\lim_{n\to 0}\sum_{\{S^\alpha,S^\beta,\dots,S^n\}}\exp\left(L_0\right) = 1$.

$$\begin{aligned}
&\log\sum_{\{S^\alpha,S^\beta,\dots,S^n\}}\exp\left(\beta J\sum_{\alpha<\beta}y^{\alpha\beta}S^\alpha S^\beta + \sqrt{\beta J_0}\sum_\alpha x^\alpha S^\alpha\right)\\
&\approx \log\sum_{\{S^\alpha,S^\beta,\dots,S^n\}}\exp\left(L_0\right) + \frac{\beta J_0}{2}\sum_{\alpha\beta}\epsilon^\alpha\epsilon^\beta\left\langle S^\alpha S^\beta\right\rangle_{L_0} + \frac{\beta^2 J^2}{2}\sum_{\alpha<\beta}\sum_{\gamma<\delta}\eta^{\alpha\beta}\eta^{\gamma\delta}\left\langle S^\alpha S^\beta S^\gamma S^\delta\right\rangle_{L_0}\\
&-\frac{\beta J_0}{2}\sum_{\alpha\beta}\epsilon^\alpha\epsilon^\beta\left\langle S^\alpha\right\rangle_{L_0}\left\langle S^\beta\right\rangle_{L_0} - \frac{\beta^2 J^2}{2}\sum_{\alpha<\beta}\sum_{\gamma<\delta}\eta^{\alpha\beta}\eta^{\gamma\delta}\left\langle S^\alpha S^\beta\right\rangle_{L_0}\left\langle S^\gamma S^\delta\right\rangle_{L_0}\\
&-\beta J\sqrt{\beta J_0}\sum_\delta\sum_{\alpha<\beta}\epsilon^\delta\eta^{\alpha\beta}\left\langle S^\delta\right\rangle_{L_0}\left\langle S^\alpha S^\beta\right\rangle_{L_0}\\
&+\beta J\sqrt{\beta J_0}\sum_\delta\sum_{\alpha<\beta}\epsilon^\delta\eta^{\alpha\beta}\left\langle S^\alpha S^\beta S^\delta\right\rangle_{L_0}.
\end{aligned}$$

Adding contributions from $-\sum_{\alpha<\beta}\frac{1}{2}\left(y^{\alpha\beta}\right)^2 - \sum_\alpha\frac{1}{2}\left(x^\alpha\right)^2$ in Eq.56, we can summarize the second-order dependence of $[f]$ on ϵ, η (up to a positive scalar βn) as

[4]Complaint is my own

$$\Delta := \frac{1}{2}\sum_{\alpha\beta}\left\{D_{\alpha\beta} - \beta J_0\left(\left\langle S^\alpha S^\beta\right\rangle_{L_0} - \left\langle S^\alpha\right\rangle_{L_0}\left\langle S^\beta\right\rangle_{L_0}\right)\right\}\epsilon^\alpha\epsilon^\beta \tag{59}$$

$$+\beta J\sqrt{\beta J_0}\sum_{\delta}\sum_{\alpha<\beta}\left(\left\langle S^\delta\right\rangle_{L_0}\left\langle S^\alpha S^\beta\right\rangle_{L_0} - \left\langle S^\alpha S^\beta S^\delta\right\rangle_{L_0}\right)\epsilon^\delta\eta^{\alpha\beta} \tag{60}$$

$$+\frac{1}{2}\sum_{\alpha<\beta}\sum_{\gamma<\delta}\left\{D_{(\alpha\beta)(\delta\gamma)} - \beta^2J^2\left(\left\langle S^\alpha S^\beta S^\gamma S^\delta\right\rangle_{L_0} - \left\langle S^\alpha S^\beta\right\rangle_{L_0}\left\langle S^\gamma S^\delta\right\rangle_{L_0}\right)\right\}\eta^{\alpha\beta}\eta^{\gamma\delta}. \tag{61}$$

Here D is the Kronecker's delta function. $D_{(\alpha\beta)(\delta\gamma)} == 1$ only if $\alpha,\beta = \delta,\gamma$. We now consider the Hessian G. Remember that $\left\langle S^\alpha\right\rangle_{L_0} = \left\langle S^\beta\right\rangle_{L_0}$etc.

$$G_{\alpha\alpha} = \frac{\partial^2}{\partial\epsilon^\alpha\partial\epsilon^\alpha}\Delta = 1 - \beta J_0\left(\left\langle S^\alpha S^\beta\right\rangle_{L_0} - \left\langle S^\alpha\right\rangle_{L_0}\left\langle S^\beta\right\rangle_{L_0}\right) = 1 - \beta J_0\left(1 - \left\langle S^\alpha\right\rangle_{L_0}^2\right) \equiv A$$

$$G_{\alpha\beta} = \frac{\partial^2}{\partial\epsilon^\alpha\partial\epsilon^\beta}\Delta - \beta J_0\left(\left\langle S^\alpha S^\beta\right\rangle_{L_0} - \left\langle S^\alpha\right\rangle_{L_0}^2\right) \equiv B$$

$$G_{(\alpha\beta)(\alpha\beta)} = \frac{\partial^2}{\partial\eta^{\alpha\beta}\partial\eta^{\alpha\beta}}\Delta = 1 - \beta^2J^2\left(1 - \left\langle S^\alpha S^\beta\right\rangle_{L_0}^2\right) \equiv P$$

$$G_{(\alpha\beta)(\alpha\gamma)} = -\beta^2J^2\left(\left\langle S^\beta S^\gamma\right\rangle - \left\langle S^\alpha S^\beta\right\rangle^2\right) \equiv Q$$

$$G_{(\alpha\beta)(\gamma\delta)} = -\beta^2J^2\left(\left\langle S^\alpha S^\beta S^\gamma S^\delta\right\rangle_{L_0} - \left\langle S^\alpha S^\beta\right\rangle_{L_0}^2\right) \equiv R$$

$$G_{\alpha(\alpha\beta)} = \frac{\partial^2}{\partial\epsilon^\alpha\partial\eta^{\alpha\beta}} = \beta J\sqrt{\beta J_0}\left(\left\langle S^\alpha\right\rangle_{L_0}\left\langle S^\alpha S^\beta\right\rangle_{L_0} - \left\langle S^\beta\right\rangle_{L_0}\right) \equiv C$$

$$G_{\gamma(\alpha\beta)} = \beta J\sqrt{\beta J_0}\left(\left\langle S^\gamma\right\rangle_{L_0}\left\langle S^\alpha S^\beta\right\rangle_{L_0} - \left\langle S^\alpha S^\beta S^\gamma\right\rangle_{L_0}\right) \equiv D$$

To compute these values, we need to calculate the correlation functions, including $\langle S\rangle_{L_0}, \left\langle S^\alpha S^\beta\right\rangle_{L_0}, \left\langle S^\alpha S^\beta S^\gamma\right\rangle_{L_0}, \left\langle S^\alpha S^\beta S^\gamma S^\delta\right\rangle_{L_0}$. Here the p-point correlation function is defined as

$$\int Dz\left(\tanh\beta\tilde{H}(z)\right)^p \tag{62}$$

In the paramagnetic phase, all $p = 1,2,3,4$ correlation functions are zeros. The only non-zero values are A and P, which are on the diagonal. Therefore the matrix has two eigenvalues, A and P. To make it positive-definite, we need $A > 0, P > 0$ and thus

$$1 - \beta J_0 > 0 \Rightarrow T > J_0$$

$$1 - \beta^2J^2 > 0 \Rightarrow T > J.$$

This condition can be shown to be always satisfied in the paramagnetic phase. Thus, the solution is stable in this phase.

In the ordered phase, $m = 0, q > 0$. Consider possible eigenvectors of G. Eigenvectors should have the form $[\epsilon^\alpha, \epsilon^\beta, ..., \epsilon^n, \eta^{\alpha\beta}, ...,]$, which is of dimension $\frac{n(n+1)}{2}$. There are three forms of possible eigenvectors.

The first form, which describes one eigenvector, has $\forall\alpha : \epsilon^\alpha = \epsilon; \forall\alpha,\beta : \eta^{\alpha\beta} = \eta$. The corresponding eigenvalue is

$$\lambda_1 = \frac{1}{2}\left\{A - B + P - 4Q + 3R \pm \sqrt{(A - B - P + 4Q - 3R)^2 - 8(C - D)^2}\right\}. \tag{63}$$

The second possible form, which describes n eigenvectors, has form $\epsilon^\theta = a$ for a specific θ, $\forall\alpha \neq \theta : \epsilon^\alpha = b$, $\alpha = \theta$ or $\beta = \theta : \eta^{\alpha\beta} = c$, $\alpha \neq \beta \neq \theta = d$. The corresponding eigenvalue approaches λ_1 as $n \to 0$.

A final form, which describes $\frac{n(n-1)}{2}$ eigenvectors, has form $\epsilon^\theta = \epsilon^\nu = a, \eta^{\theta\nu} = c, \eta^{\theta\alpha} = \eta^{\nu\alpha} = d, \eta^{\alpha\beta} = e$. Its eigenvalue is $\lambda_3 = P - 2Q + R$.

We can be sure that any eigenvector is in one of these forms by confirming that they define $\frac{n(n+1)}{2}$ eigenvectors in total, which matches the number of eigenvectors for the matrix.

To make sure $\lambda_1 > 0$, a sufficient (but not necessary) condition is that $A - B > 0$ and $P - 4Q + 3R > 0$. Since

$$
\left(\begin{array}{ccc|ccc}
0 & q_1 & q_1 & & & \\
q_1 & 0 & q_1 & & q_0 & \\
q_1 & q_1 & 0 & & & \\
\hline
 & & & 0 & q_1 & q_1 \\
 & q_0 & & q_1 & 0 & q_1 \\
 & & & q_1 & q_1 & 0
\end{array}\right).
$$

Figure 1: 1RSB q matrix.

$$A - B \propto \frac{\partial^2 [f]}{\partial m^2} > 0, P - 4Q + 3R \propto -\frac{\partial^2 [f]}{\partial q^2} > 0$$

(the first is true because m minimizes the free energy; the second is true because q maximizes the free energy due to issues introduced by the replica method when $n < 1$). Thus, $\lambda_1 > 0$ is always true. Since the second eigenvalue approaches this one, it is also positive. For the last eigenvalue to be positive, we need

$$P - 2Q + R = 1 - \beta^2 J^2 (1 - 2q + r) > 0$$

where r is the 4-point correlation as defined in Eq.62. This leads to

$$\left(\frac{T}{J}\right)^2 > \int D z \operatorname{sech}^4 \left(\beta J \sqrt{q} z + \beta J_0 m\right). \tag{64}$$

This equation is solved numerically to give the boundary where the symmetry is broken. The boundary on $J_0 - T$ plane is termed the de Almeida-Thouless (AT) line.

5 The Parisi solution

5.1 Multi-step replica symmetry breaking (RSB)

Under the replica symmetry assumption, $q^{\alpha\beta}$ do not depend on α, β, and neither does m^α. To break the symmetry, we need to consider structures in the q matrix. Recall that in the replica symmetry assumption, $\forall \alpha < \beta : q_{\alpha\beta} = q$. We now define a n-RSB matrix. Consider 1-RSB. We introduce an integer $m_1 \leq n$. As an example, consider $n = 6$ and $m_1 = 3$. Then the q matrix would look like Fig. 1.

We can perform the same procedure iteratively, introducing $m_2, m_3 ...$, and $q_2, q_3, ...$ in the process (See Fig. 3.2 in Nishimori). They are all integers and should satisfy

$$n \geq m_1 \geq m_2 \geq ... \geq 1. \tag{65}$$

We now define a function

$$q(x) = q_i \quad (m_{i+1} \leq x \leq m_i). \tag{66}$$

Clearly this function is defined on non-integer values. We now take the non-Kosher limit of $n \to 0$, and thus all the m go to 0. And then magically at this limit, we consider $0 \leq m_1 \leq m_2 .. \leq 1$ (since all the m move to the left of 1 with n I guess...?).

Also note that we maintain our replica symmetry assumption for m^α.

5.2 First step RSB

To calculate the free energy right now, we need to go back to the expression for $[Z^n]$. Consider the function L defined in Eq.18. Since we are considering $J_0 = h = 0$, the only thing left to compute is

$$\sum_{\alpha<\beta} q_{\alpha\beta} S^\alpha S^\beta = -\frac{1}{2}\left\{ q_0 \left(\sum_{\alpha}^{n} S^\alpha\right)^2 + (q_1 - q_0) \sum_{b=1}^{n/m_1} \left(\sum_{\alpha\in B_b}^{m_1} S^\alpha\right)^2 - nq_1 \right\}. \tag{67}$$

Here B_b denotes the b-th block, of which there are n/m_1 ones. The first term in the bracket is like filling the q-matrix with q_0; the second term is like replacing all the q_0 inside blocks to q_1; the last term removes the diagonal of the matrix. Similarly, the term in the free energy expression Eq.28 is

$$\lim_{n\to 0} \frac{1}{n} \sum_{\alpha\neq\beta} q_{\alpha\beta}^2 = \lim_{n\to 0} \frac{1}{n} \left\{ n^2 q_0^2 + \frac{n}{m_1} m_1^2 \left(q_1^2 - q_0^2\right) - nq_1^2 \right\} = (m_1 - 1)\, q_1^2 - m_1 q_0^2. \tag{68}$$

For the reader's convenience I copy Eq.28 as

$$\beta[f_{1RSB}] = -\frac{[\log Z]}{N} = -\lim_{n\to 0} \frac{[Z^n] - 1}{Nn} = \lim_{n\to 0} \left\{ -\frac{\beta^2 J^2}{4} + \frac{\beta^2 J^2}{2n} \sum_{\alpha<\beta} \left(q_{\alpha\beta}^\star\right)^2 + \frac{\beta J_0}{2n} \sum_{\alpha} \left(m_\alpha^\star\right)^2 - \frac{1}{n} \log \sum_{\{S^\alpha, S^\beta, \ldots, S^n\}} \exp\left(L_{1RSB}\right) \right\}. \tag{69}$$

First, plug in Eq.68 and taking the limit for the first three terms

$$\beta[f_{1RSB}] = \frac{\beta^2 J^2}{4} \left\{ (m_1 - 1)\, q_1^2 - m_1 q_0^2 - 1 \right\} + \frac{\beta J_0}{2} m^2 - \frac{1}{n} \log \sum_{\{S^\alpha, S^\beta, \ldots, S^n\}} \exp\left(L_{1RSB}\right). \tag{70}$$

We then plug Eq.67 into the expression for L (copied below for convenience) to get

$$L\left(\{q_{\alpha\beta}, m_\alpha\}\right) := \beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} S^\alpha S^\beta + \beta \sum_{\alpha} \left(J_0 m_\alpha + h\right) S^\alpha. \tag{71}$$

$$\Rightarrow L_{1RSB} = \frac{\beta^2 J^2}{2} \left\{ q_0 \left(\sum_{\alpha}^{n} S^\alpha\right)^2 + (q_1 - q_0) \sum_{b=1}^{n/m_1} \left(\sum_{\alpha\in B_b}^{m_1} S^\alpha\right)^2 - nq_1 \right\} + \beta \sum_{\alpha} \left(J_0 m_\alpha + h\right) S^\alpha. \tag{72}$$

Our goal is again to linearize L with respect to S^α. When we were doing the replica symmetric calculations, we introduced one Gaussian variable to linearize the exponent (see Eq.38). There we only have to deal with $\left(\sum_\alpha^n S^\alpha\right)^2$ and thus introducing one variable is enough. Here we have both $\left(\sum_\alpha^n S^\alpha\right)^2$, and $n/m_1 \left(\sum_{\alpha\in B_b}^{m_1} S^\alpha\right)^2$. So in total we introduce $1 + n/m_1$ Gaussian variables. First, introduce a Gaussian variable u to linearize $\left(\sum_\alpha^n S^\alpha\right)^2$.

$$\begin{aligned}
&\frac{1}{n} \log \sum_{\{S^\alpha, S^\beta, \ldots, S^n\}} \exp\left(L_{1RSB}\right) \\
&= \frac{1}{n} \log \sum_{\{S^\alpha, S^\beta, \ldots, S^n)} \int Du \exp\left(\beta J \sqrt{q_0} u \sum_{\alpha} S^\alpha - \frac{n}{2} \beta^2 J^2 q_0 + \sum S^\alpha \beta \left(J_0 m + h\right) + (q_1 - q_0) \sum_{b=1}^{n/m_1} \left(\sum_{\alpha\in B_b}^{m_1} S^\alpha\right)^2 \right) \\
&= -\frac{1}{2} \beta^2 J^2 q_0 + \underbrace{\frac{1}{n} \log \sum_{\{S^\alpha, S^\beta, \ldots, S^n)} \int Du \exp\left(\sum S^\alpha \left(\beta J \sqrt{q_0} u + \beta \left(J_0 m + h\right)\right) + (q_1 - q_0) \sum_{b=1}^{n/m_1} \left(\sum_{\alpha\in B_b}^{m_1} S^\alpha\right)^2 \right)}_{\Delta}
\end{aligned}$$

This step is exactly the same as that for linearizing $\left(\sum_\alpha^n S^\alpha\right)^2$ in the symmetric calculations, except that q is replaced by q_0 here, and that we denoted the new variable u instead of z. As it is the case before, the $-\frac{n}{2}\beta^2 J^2 q$ term came from $\left(\sum_\alpha S^\alpha\right)^2 - 2\sum_{\alpha<\beta} S^\alpha S^\beta = n$. We do not create an analogous term when we linearize $\sum_{b=1}^{n/m_1} \left(\sum_{\alpha\in B_b}^{m_1} S^\alpha\right)^2$ because they are already written in the form of $\left(\sum_\alpha S^\alpha\right)^2$.

Now, introduce a separate v_b for each $\left(\sum_{\alpha\in B_b}^{m_1} S^\alpha\right)^2$ to obtain an expression for the second term above as

$$\frac{1}{n}\log\sum_{\{S^\alpha,S^\beta,\ldots,\}}\int Du\prod_{b=1}^{n/m_1}\left(\int Dv_b\right)\exp\left(\sum_\alpha S^\alpha\beta J\sqrt{q_0}u+\sum_\alpha S^\alpha\beta\left(J_0m+h\right)+\beta J\sqrt{q_1-q_0}\sum_{b=1}^{n/m_1}v_b\sum_{\alpha\in B_b}S^\alpha\right) \tag{73}$$

$$=\frac{1}{n}\log\int Du\sum_{\{S^\alpha,S^\beta,\ldots,S^n\}}\prod_{b=1}^{n/m_1}\left\{\int Dv\exp\left(\sum_\alpha S^\alpha\beta\left(J_0m+h+\beta J\sqrt{q_0}u\right)\right)\exp\left(\beta J\sqrt{q_1-q_0}v\sum_{\alpha\in B_b}^{m_1}S^\alpha\right)\right\} \tag{74}$$

$$=\frac{1}{n}\log\int Du\prod_{b=1}^{n/m_1}\left\{\int Dv_b\sum_{\{S^\alpha\}\in B_b}\exp\left(\sum_{\{S^\alpha\}\in B_b}\beta\left(J_0m+h+\beta J\sqrt{q_0}u+J\sqrt{q_1-q_0}v_b\right)\right)\right\} \tag{75}$$

$$=\frac{1}{n}\log\int Du\prod_{b=1}^{n/m_1}\left\{\int Dv_b\left\{\sum_{S=\pm1}\exp\left(S\beta\left(J_0m+h+\beta J\sqrt{q_0}u+J\sqrt{q_1-q_0}v\right)\right)\right\}^{m_1}\right\} \tag{76}$$

$$=\frac{1}{n}\log\int Du\left\{\int Dv\left\{\sum_{S=\pm1}\exp\left(S\beta\left(J_0m+h+\beta J\sqrt{q_0}u+J\sqrt{q_1-q_0}v\right)\right)\right\}^{m_1}\right\}^{n/m_1} \tag{77}$$

$$=\frac{1}{n}\log\int Du\left\{\int Dv\left\{2\cosh\left(\beta\left(J_0m+h+\beta J\sqrt{q_0}u+J\sqrt{q_1-q_0}v\right)\right)\right\}^{m_1}\right\}^{n/m_1} \tag{78}$$

$$=\frac{1}{n}\log\int Du\exp\left\{\frac{n}{m_1}\log\left\{\int Dv\left\{2\cosh\left(\beta\left(J_0m+h+\beta J\sqrt{q_0}u+J\sqrt{q_1-q_0}v\right)\right)\right\}^{m_1}\right\}\right\} \tag{79}$$

$$\approx\frac{1}{n}\log\left\{1+\frac{n}{m_1}\int Du\log\left\{\int Dv\left\{2\cosh\left(\beta\left(J_0m+h+\beta J\sqrt{q_0}u+J\sqrt{q_1-q_0}v\right)\right)\right\}^{m_1}\right\}\right\}\quad\text{expand exponent around 0} \tag{80}$$

$$\approx\frac{1}{n}\frac{n}{m_1}\int Du\log\left\{\int Dv\left\{2\cosh\left(\beta\left(J_0m+h+\beta J\sqrt{q_0}u+J\sqrt{q_1-q_0}v\right)\right)\right\}^{m_1}\right\}\quad\text{expand log around 1} \tag{81}$$

$$=\log 2+\frac{1}{m_1}\int Du\log\left\{\int Dv\left\{\cosh\left(\beta\left(J_0m+h+\beta J\sqrt{q_0}u+J\sqrt{q_1-q_0}v\right)\right)\right\}^{m_1}\right\}\quad\text{pull the factor of 2 out} \tag{82}$$

[5]Letting

$$\Xi=\beta\left(J_0m+h+\beta J\sqrt{q_0}u+J\sqrt{q_1-q_0}v\right). \tag{86}$$

We have

$$\Delta=\log 2+\frac{1}{m_1}\int Du\log\left\{\int Dv\left\{\cosh\Xi\right\}^{m_1}\right\}. \tag{87}$$

Plugging this back into the expression for $\beta[f_{1RSB}]$ to get

$$\beta[f_{1RSB}]=\frac{\beta^2J^2}{4}\left\{\left(m_1-1\right)q_1^2-m_1q_0^2+2q_1-1\right\}+\frac{\beta J_0}{2}m^2-\log 2-\frac{1}{m_1}\int Du\log\left\{\int Dv\left\{\cosh\Xi\right\}^{m_1}\right\}. \tag{88}$$

This is now a function of q_0, q_1, m, m_1 (m is the magnetization and m_1 is the block size!). We now need arg max for all four variables. Differentiating $\beta[f_{1RSB}]$ w.r.t. them and setting the derivatives to zero to get

$$m^*=\int Du\frac{\int Dv\left\{\cosh\Xi\right\}^{m_1}\tanh\Xi}{\int Dv\left\{\cosh\Xi\right\}^{m_1}} \tag{89}$$

---

[5]For the last few lines, it may be useful to compare to the symmetric calculations below
Hence the last term in Eq.36 can be written as

$$\frac{1}{n}\log\int Dz\exp\left(n\log 2\cosh\left(\beta J\sqrt{q}z+\beta\left(J_0m+h\right)\right)-\frac{n}{2}\beta^2J^2q\right). \tag{83}$$

We now take the small n limit, and expand the exponential around 0

$$\approx\frac{1}{n}\log\left\{1+n\int Dz\log 2\cosh\left(\beta J\sqrt{q}z+\beta\left(J_0m+h\right)\right)-\frac{n}{2}\beta^2J^2q\right\} \tag{84}$$

Expand the logarithm around 1 to get

$$\int Dz\log 2\cosh\left(\beta J\sqrt{q}z+\beta\left(J_0m+h\right)\right)-\frac{n}{2}\beta^2J^2q. \tag{85}$$

$$q_0^* = \int Du \left(\frac{\int Dv \{\cosh \Xi\}^{m_1} \tanh \Xi}{\int Dv \{\cosh \Xi\}^{m_1}} \right)^2 \tag{90}$$

$$q_1^* = \int Du \frac{\int Dv \{\cosh \Xi\}^{m_1} \{\tanh \Xi\}^2}{\int Dv \{\cosh \Xi\}^{m_1}}. \tag{91}$$

Apparently, the condition for m_1 is not important (since we consider multi-step RSB anyways) It can be verified, by again computing the Hessian of $\beta[f_{1RSB}]$ w.r.t. these four parameters that negative entropy occurs again at low temperatures.

6 Thouless-Anderson-Palmer (TAP) equations

The TAP equations are approximate equations of state for the local magnetization m_i for every i. One way to derive them is through Plefka expansions. Recall definition of the free energy (for a specific J)

$$F(\boldsymbol{s}) = -T \log Z(\boldsymbol{s}) = -T \log \sum_{\{\boldsymbol{s}\}} \exp(-\beta H(\boldsymbol{s})). \tag{92}$$

Consider the constrained optimization problem where we'd like to minimize $H(\boldsymbol{s})$ while specifying the thermal averages of $\boldsymbol{s}$ (i.e. the magnetization). We obtain a surrogate Hamiltonian (i.e. the Lagrangian)

$$\tilde{H}(\boldsymbol{s}, \alpha, \boldsymbol{m}) = \alpha H(\boldsymbol{s}) - \sum_i \lambda_i (S_i - m_i)$$

where the $\{\lambda_i\}$ are Lagrange multipliers enforcing the constraints. The surrogate free energy becomes

$$\begin{aligned}
\tilde{F} &= -T \log \sum_{\{\boldsymbol{s}\}} \exp\left(-\beta \tilde{H}(\boldsymbol{s}, \alpha, \boldsymbol{m})\right) && (93)\\
&= -T \log \sum_{\{\boldsymbol{s}\}} \exp\left(-\alpha\beta H(\boldsymbol{s}) + \beta \sum_i \lambda_i S_i - \beta \sum_i \lambda_i m_i\right) && (94)\\
&= -T \log \left\{ \exp\left(-\beta \sum_i \lambda_i m_i\right) \sum_{\{\boldsymbol{s}\}} \exp\left(-\alpha\beta H(\boldsymbol{s}) + \beta \sum_i \lambda_i S_i\right) \right\} && (95)\\
&= -T \log \sum_{\{\boldsymbol{s}\}} \exp\left(-\alpha\beta H(\boldsymbol{s}) + \beta \sum_i \lambda_i S_i\right) - \sum_i \lambda_i m_i. && (96)
\end{aligned}$$

The idea here is the following. The true Hamiltonian would have $\alpha = 1$. We expand $\tilde{F}$ around $\alpha = 0$, and then plug in $\alpha = 1$ to get

$$\tilde{F}(\alpha = 1) \approx \tilde{F}(\alpha = 0) + \left.\frac{d\tilde{F}}{d\alpha}\right|_{\alpha=0} + \frac{1}{2} \left.\frac{d^2\tilde{F}}{d\alpha^2}\right|_{\alpha=0}. \tag{97}$$

Denote $\langle \cdot \rangle_{\tilde{H}}$ as the thermal average with $\tilde{H}$ as the Hamiltonian.

$$\begin{aligned}
\tilde{F}(\alpha = 0) &= -T \log \sum_{\{\boldsymbol{s}\}} \exp\left(\beta \sum_i \lambda_i S_i - \beta \sum_i \lambda_i m_i\right)\\
&= \sum_i \lambda_i m_i - T \log \sum_{\{\boldsymbol{s}\}} \exp\left(\beta \sum_i \lambda_i S_i\right)\\
&= \sum_i \lambda_i m_i - T \log \sum_{\{\boldsymbol{s}\}} \exp\left(\beta \sum_i \lambda_i S_i\right)\\
&= \sum_i \lambda_i m_i - T \log \prod_{i=1} 2\cosh(\beta\lambda_i)\\
&= \sum_i \lambda_i m_i - T \log \prod_{i=1} 2\cosh(\beta\lambda_i)
\end{aligned}$$

We now solve for the $\{\lambda_i\}$ from the constraint equations $\{m_i = \langle S_i \rangle_{\tilde{H}}\}$. Since the only dependence of $\tilde{H}$ on S_i at $\alpha = 0$ is through $-\lambda_i S_i$

$$m_i = \langle S_i \rangle_{\tilde{H}} = \frac{e^{\beta\lambda_i} - e^{-\beta\lambda_i}}{e^{\beta\lambda_i} + e^{-\beta\lambda_i}} = \tanh(\beta\lambda_i) \tag{98}$$

$$\Rightarrow \lambda_i = T \tanh^{-1} m_i = \frac{T}{2} \left(\log(1+m_i) - \log(1-m_i)\right). \tag{99}$$

Then

$$\cosh(\beta\lambda_i) = \cosh\left(\tanh^{-1}(m_i)\right) = \frac{1}{\sqrt{1-m_i^2}} \tag{100}$$

$$\begin{aligned}
\tilde{F}(\alpha=0) &= \frac{T}{2}\sum_i m_i \left(\log(1+m_i) - \log(1-m_i)\right) - T\sum_i \log\frac{2}{\sqrt{1-m_i^2}} && (101)\\
&= \frac{T}{2}\sum_i m_i \left(\log(1+m_i) - \log(1-m_i)\right) - T\sum_i \log 2 + \frac{T}{2}\log\left(1-m_i^2\right) && (102)\\
&= \frac{T}{2}\sum_i m_i \left(\log(1+m_i) - \log(1-m_i)\right) - T\sum_i \log 2 + \frac{T}{2}\left(\log(1-m_i) + \log(1+m_i)\right) && (103)\\
&= \frac{T}{2}\sum_i \left\{m_i \log(1+m_i) - m_i\log(1+m_i) + \log(1+m_i) + \log(1-m_i) - 2\log 2\right\} && (104)\\
&= \frac{T}{2}\sum_i \left\{(1+m_i)\log(1+m_i) + (1-m_i)\log(1-m_i) - (1+m_i+1-m_i)\log 2\right\} && (105)\\
&= T\sum_i \left\{\frac{1+m_i}{2}\log\left(\frac{1+m_i}{2}\right) + \frac{1-m_i}{2}\log\left(\frac{1-m_i}{2}\right)\right\} && (106)
\end{aligned}$$

$$\left.\frac{\partial\tilde{F}}{\partial\alpha}\right|_{\alpha=0} = \left.\langle H(\boldsymbol{s})\rangle_{\tilde{H}}\right|_{\alpha=0} = -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j - \sum_i h_i m_i \tag{107}$$

$$\left.\frac{\partial^2\tilde{F}}{\partial\alpha^2}\right|_{\alpha=0} = -\beta\left.\left\langle H\left(H - \langle H\rangle_{\tilde{H}} - \sum_i \frac{\partial\lambda_i}{\partial a}(S_i - m_i)\right)\right\rangle_{\tilde{H}}\right|_{\alpha=0} \tag{108}$$

$$\left.\frac{\partial\lambda_i}{\partial a}\right|_{\alpha=0} = \left.\frac{\partial}{\partial a}\frac{\partial\tilde{F}}{m_i}\right|_{\alpha=0} = -\sum_{j\neq i} J_{ij} m_j. \tag{109}$$

$$\begin{aligned}
\left.\frac{\partial^2\tilde{F}}{\partial\alpha^2}\right|_{\alpha=0} &= -\beta\left.\left\langle H\left(H - \langle H\rangle_{\tilde{H}} + \sum_i\sum_{j\neq i} J_{ij} m_j (S_i - m_i)\right)\right\rangle_{\tilde{H}}\right|_{\alpha=0} && (110)\\
&= -\beta\left.\left\langle H\left(H + \frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j - \sum_{i\neq j} J_{ij} m_i m_j + \sum_{i\neq j} J_{ij} S_i m_j\right)\right\rangle_{\tilde{H}}\right|_{\alpha=0} && (111)\\
&= -\beta\left\{\mathrm{var}(H)_{\tilde{H}} + \left\langle -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j + H\sum_{i\neq j} J_{ij} S_i m_j\right\rangle_{\tilde{H}}\right\} && (112)
\end{aligned}$$

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